Cumulative Reasoning with Large Language Models

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Abstract

Despite the recent advancements in language models (LMs), their ability to solve complex problems remains limited. This paper introduces Cumulative Reasoning (CR), a novel approach that utilizes LMs cumulatively and iteratively, mirroring human thought processes for problem-solving. CR decomposes tasks into smaller, manageable components and leverages previous propositions for effective composition, significantly enhancing problem-solving capabilities. We demonstrate CR's superiority through several complex reasoning tasks: it outperforms existing methods in logical inference tasks with up to a 9.3% improvement, achieving 98.04% accuracy on the curated FOLIO wiki dataset. In the Game of 24, it achieves 98% accuracy, marking a 24% improvement over the prior state-of-the-art. Additionally, CR sets new state-of-the-art on the MATH dataset, achieving a 4.2% increase from previous methods and a 43% relative improvement in the most challenging problems. By extending CR to incorporate a code environment without external aids like retrieval or web browsing, we further harness the computational and logical reasoning capabilities of LMs, achieving a remarkable 72.2% accuracy on the MATH dataset and outperforming the PAL/PoT method by 38.8%. Our work not only sets new state-of-the-art but also paves the way toward more sophisticated AI reasoning methods‡.

1 Introduction

Despite the remarkable advances made by large language models (LLMs) in a variety of applications [3, 9, 40–42, 44], they still struggle to provide stable and accurate answers when faced with highly complex tasks. For instance, it has been observed that language models have difficulty directly generating correct answers for high school math problems [29].

Drawing from Kahneman's dual-process theory [24], which distinguishes between fast, intuitive thought (System 1) and slower, more deliberate thought (System 2), current LLMs are predominantly aligned with System 1, which restricts their ability to engage in the systematic and logical reasoning required for complex problem-solving tasks.

Recent efforts to bridge this gap include Chain-of-Thought (CoT) prompting [58] and Tree-of-Thought (ToT) methodologies [32, 63], which guide LLMs through a more structured reasoning process. However, these

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‡The code is available at `https://github.com/iiis-ai/cumulative-reasoning`.

approaches lack mechanisms for dynamically storing and leveraging intermediate results, a crucial aspect of human cognitive processes.

In this work, we introduce Cumulative Reasoning (CR), a novel framework that characterizes a more holistic representation of the thinking process. CR orchestrates a symphony of three LLM roles—the proposer, verifier(s), and reporter—to iteratively propose, validate, and compile reasoning steps into a comprehensive solution. This decomposition and composition strategy effectively transforms complex, multifaceted problems into a series of manageable tasks, significantly enhancing the problem-solving capabilities of LLMs.

Our empirical evaluation spans three distinct areas:

1. Logical Inference Tasks: CR demonstrates superior performance on datasets like FOLIO wiki and AutoTNLI, with improvements of up to 9.3% and an outstanding 98.04% accuracy on a curated version of the FOLIO dataset.
2. The Game of 24: We achieved 98% accuracy, marking a 24% improvement over the existing state-of-the-art method ToT [63] while using only about 25% visited states.
3. Solving MATH problems: CR establishes new benchmarks with a margin of 4.2% over previous methods [14, 67] without external tools. Noteworthy, our method achieves notable 43% relative improvements on the hardest level 5 problems (22.4% $\rightarrow$ 32.1%). Moreover, by integrating CR with a Python code environment—absent external aids like retrieval systems, we achieves a 72.2% accuracy on the MATH dataset, outperforming previous methods such as PoT [4] and PAL [16] with 38.8% relative improvement and demonstrating the adaptability and robustness of CR across a spectrum of complex tasks.

Through these contributions, we not only advance the state-of-the-art in problem-solving with language models but also provide a versatile framework inspired by higher-order logic that approximates human-like reasoning.

2 Preliminaries

2.1 Logic

Propositional logic, the most fundamental system of logic, encompasses elements p, q, r and a variety of operations. These include "and" ($p \wedge q$), "or" ($p \vee q$), "implies" ($p \Rightarrow q$), and "not" ($\neg p$). The constants true and false are denoted as 1 and 0 respectively. This system adheres to the following rules:

$$x \wedge x = x, \quad x \vee x = x, \quad 1 \wedge x = x, \quad 0 \vee x = x, \quad x \wedge (y \vee x) = x = (x \wedge y) \vee x.$$

and distributive laws:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z).$$

In a Boolean algebra, every element x has a complement $\neg x$ and the following holds true:

$$x \wedge \neg x = 0, \quad x \vee \neg x = 1, \quad \neg\neg x = x.$$

Building upon propositional logic, first-order logic (FOL) introduces universal quantification ($\forall$) and existential quantification ($\exists$) to describe more intricate propositions. For instance, the statement "$\forall_x \text{Dog}(x) \Rightarrow \text{Animal}(x)$" translates to "for every x, if x is a dog, then it is also an animal". Higher-order logic (HOL) represents a sophisticated formalism that permits quantification over functions and predicates, an ability that contrasts sharply with FOL, which restricts quantification to individual objects. For a detailed discussion on the distinctive characteristics of HOL, as opposed to FOL, please refer to Appendix D.1.

2.2 Illustrative example

Consider the following example adapted from the FOLIO dataset [19], where empirically only the text statements (excluding logical propositions) will be given as context:

1. All monkeys are mammals: $\forall x(\text{Monkey}(x) \Rightarrow \text{Mammals}(x))$.
2. An animal is either a monkey or a bird: $\forall x(\text{Animal}(x) \Rightarrow (\text{Monkey}(x) \vee \text{Bird}(x)))$.
3. All birds fly: $\forall x(\text{Bird}(x) \Rightarrow \text{Fly}(x))$.
4. If something can fly, then it has wings: $\forall x(\text{Fly}(x) \Rightarrow \text{Wings}(x))$.
5. Rock is not a mammal, but Rock is an animal: $\neg\text{Mammal}(\text{Rock}) \wedge \text{Animal}(\text{Rock})$.

The question is: Does rock have wings? We have the following derivations:

a. The contrapositive of (1) is: $\forall x(\neg\text{Mammals}(x) \Rightarrow \neg\text{Monkey}(x))$.

b. (a) and (5) $\Rightarrow \neg\text{Monkey}(\text{Rock}) \wedge \text{Animal}(\text{Rock})$.

c. (2) and (5) $\Rightarrow (\text{Monkey}(\text{Rock}) \vee \text{Bird}(\text{Rock}))$

d. (b) and (c) $\Rightarrow \text{Bird}(\text{Rock})$.

e. (3) and (d) $\Rightarrow \text{Fly}(\text{Rock})$.

f. (4) and (e) $\Rightarrow \text{Wings}(\text{Rock})$.

While the observable derivation can be treated as a general "chain of thought" from (a) to (f), its internal structure is neither a chain nor a tree. Instead, it is a directed acyclic graph (DAG), with each directed edge as one step of derivation. For examples of higher-order logic, see Appendix D.1.

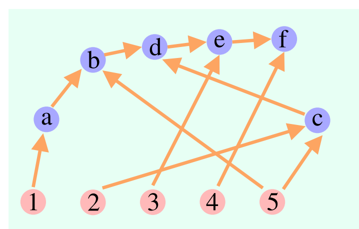

Figure 1: Illustration of our logical derivation

3 Cumulative Reasoning

3.1 Cumulative Reasoning (CR)

CR introduces a novel framework leveraging three specialized types of Large Language Models (LLMs) in a collaborative reasoning process:

1. Proposer: Suggests potential steps based on the current context, initiating the reasoning cycle.
2. Verifier(s): Assess the proposer's suggestions for accuracy, incorporating valid steps into the ongoing context.
3. Reporter: Determines the appropriate moment to conclude the reasoning process, based on whether the accumulated context leads to a definitive solution.

Our approach is visualized in Figure 2, illustrating how CR iteratively constructs and refines a solution from initial propositions to a final conclusion. In practical terms, the proposer is ideally a model pre-trained on related derivation tasks, while verifiers translate these proposals into formal systems for validation, employing either symbolic reasoning systems or incorporating a code environment.

While specialized models offer optimal performance, the flexibility of CR permits effective deployment using general-purpose LLMs like GPT-4, tailored through role-specific prompting (Appendix A details the prompt design). Notice that in our method, we introduced several different LLMs with fresh eyes by managing the thinking context of each role, beyond the self-verification capabilities of language models.

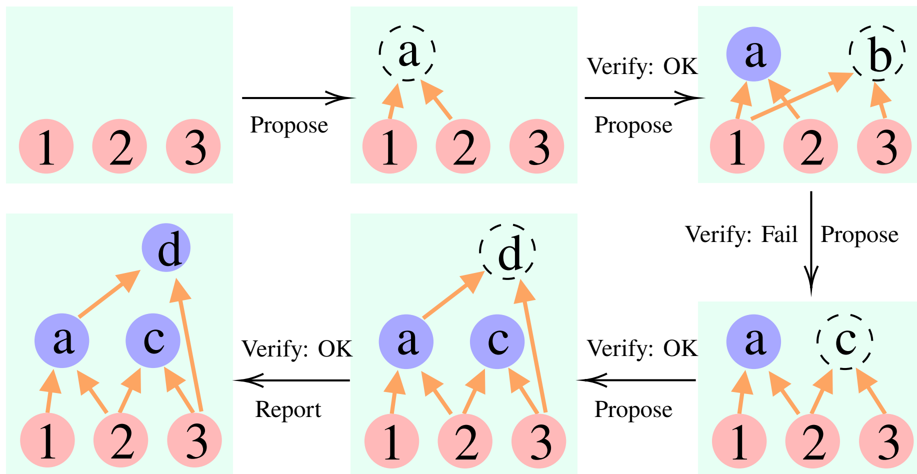

Figure 2: An illustration of Cumulative Reasoning (CR) for a problem with three premises.

The underlying rationale for CR draws from intuitionistic logic and the philosophy of mathematical constructivism—asserting that a cumulative, constructive approach is inherently suited for complex reasoning tasks. This methodology not only allows for the dynamic adjustment of the reasoning trajectory based on intermediate validations but also significantly enhances the problem-solving efficacy of LLMs.

3.2 Comparison with CoT and ToT

While superficially similar to Chain-of-Thought (CoT) and Tree-of-Thought (ToT), CR distinguishes itself through its capability to dynamically store and utilize all historically validated reasoning results to perform composition, forming a Directed Acyclic Graph (DAG) rather than a mere sequence or tree. This structural flexibility enables CR to tackle more intricate problems by leveraging a broader context of verified propositions, thus overcoming the limitations of CoT and ToT in managing complex reasoning tasks.

CR 's superiority is rooted in its synergistic integration of proposer, verifier(s), and reporter roles within a coherent framework, hence introducing fresh eyes into the reasoning process, and optimizing the accumulation and validation of intermediate results. This integrative approach fosters a deeper, more precise reasoning process that is both adaptable and error-tolerant, mirroring the nuanced and iterative nature of human problem-solving. Regarding computational complexity, please refer to Appendix B.1 for comprehensive experiments comparing different methods which shows the superiority of CR compared to CoT, CoT-SC, and ToT on several logical inference tasks including LogiQA [31] and ProofWriter [50] datasets. For a detailed quantitative theoretical analysis, please refer to Appendix C.

4 Experiments

Our experiments are based on the Microsoft guidance library [34], which offers the flexibility to intertwine generation, prompting, and logical control in a seamless flow that aligns with language models. We consider the following LLMs: GPT-3.5-turbo, GPT-4, LLaMA-13B and LLaMA-65B.

Our Proposer, Verifier(s), and Reporter in CR are implemented using the same LLM with different few-shot prompts. This approach ensures a broad application scope and simplifies implementation. We denote n as the number of generated intermediate propositions, and k as the number of majority voting times. We set the temperature $t = 0.1$ by default and $t = 0.7$ for majority voting. We also remark that both GPT-3.5-turbo and GPT-4 operate as chat-format APIs from OpenAI.

4.1 FOLIO wiki

FOLIO dataset [19] is a first-order logical inference dataset for reasoning in natural language. The label of each problem can be "True", "False", or "Unknown". See Figure 3 for an example. We observed that while the Chain-of-Thought reasoning process can generate useful intermediary results, it tends to flounder midway, failing to arrive at the correct conclusion. Conversely, the CR initially spawns two beneficial propositions and leverages them to successfully solve the problem at hand. For a deeper dive into specific examples of the FOLIO dataset, we refer to Appendix E.1.

[Problem Description]

- Example ID: 687
- **Premises**:
  1. It is not true that some giant language models do not have good performance.
  2. All language models with good performance are used by some researchers.
  3. If a language model is used by some researchers, it is popular.
  4. If BERT is a giant language model, then GPT-3 is also a giant language model.
  5. BERT is a giant language model.
- **Hypothesis**: GPT-3 is popular.
- **Label**: **[True]**

[Chain-of-Thought Reasoning by GPT-4]

- **Reasoning**: "From the premises, we have:

$$
\begin{aligned}
&1.\neg(\exists x)(\text{Giant}(x) \wedge \neg\text{GoodPerformance}(x))\\
&2.\forall x(\text{GoodPerformance}(x) \rightarrow \text{UsedByResearchers}(x))\\
&3.\forall x(\text{UsedByResearchers}(x) \rightarrow \text{Popular}(x))\\
&4.\text{BERT} \rightarrow \text{GPT-3}\\
&5.\text{BERT}
\end{aligned}
$$

  Now we know that BERT is a giant language model, so GPT-3 is also a giant language model (from premises 4 and 5). However, the premises do not provide information about GPT-3's performance or if it is used by researchers. Therefore, we cannot conclude if GPT-3 is popular."
- **Prediction**: **[Unknown]** (**Wrong**)

[Cumulative-Reasoning by GPT-4]

- **Generated Propositions**:
  1. "GPT-3 is a giant language model with good performance."
  2. "GPT-3 is used by some researchers."
- **Reasoning**: "We know that GPT-3 is a giant language model with good performance, and it is used by some researchers. According to the premises, if a language model is used by some researchers, it is popular. Therefore, GPT-3 is popular."
- **Prediction**: **[True]** (**Correct**)

Figure 3: An example from the FOLIO dataset, with solutions generated by CoT and CR. CoT will generate the answer directly through a chain of thought. By contrast, CR will first generate a few propositions, and conclude with the help of the generated propositions.

The FOLIO dataset is a composite of 1435 examples, wherein 52.5% of these instances have been crafted drawing upon knowledge from randomly selected Wikipedia pages. This approach guarantees the infusion of abundant linguistic variations and a rich vocabulary within the corpus. The residual 47.5% of the examples have been penned in a hybrid style, rooted in a variety of complex logical templates. Acknowledging that contemporary LLMs are pre-trained on a considerable volume of human-written corpus, we direct our experiments towards those examples derived from Wikipedia, hereby referred to as FOLIO-wiki. Once a handful of examples are moved aside for few-shot prompts and those examples without source labels for validations are excluded, we are left with a testable collection of 534 examples.

Our experimental design employs the LLaMA base model and GPT APIs directly, circumventing the need for fine-tuning with logical inference datasets and thus ensuring a faithful comparison. The results, displayed in Table 1, reveal that CR consistently surpasses Direct (standard Input-Output prompt), CoT, and CoT-SC, with a performance margin spanning up to 8.42%. Notably, GPT-4 paired with Cumulative Reasoning (CR) achieves an accuracy rate of 87.45%, outperforming GPT-4 with CoT-SC, which reports an accuracy rate of 85.02%. For more experiments on LogiQA [31], ProofWriter [50], and LogicalDeduction datasets [48] and more ablation studies, please refer to Appendix B.

Table 1: Results for various reasoning approaches on FOLIO-wiki dataset.

Model	Method	Acc. ↑ (%)
-	[Random]	33.33
LLaMA-13B	Direct	44.75
	CoT	49.06 (+4.31)
	CoT-SC (k = 16)	<u>52.43</u> (+7.68)
	CR (**ours**, n = 2)	**53.37** (**+8.62**)
LLaMA-65B	Direct	67.42
	CoT	67.42 (+0.00)
	CoT-SC (k = 16)	<u>70.79</u> (+3.37)
	CR (**ours**, n = 2)	**72.10** (**+4.68**)
GPT-3.5-turbo	Direct	62.92
	CoT	<u>64.61</u> (+1.69)
	CoT-SC (k = 16)	63.33 (+0.41)
	CR (**ours**, n = 2)	**73.03** (**+10.11**)
GPT-4	Direct	80.52
	CoT	84.46 (+3.94)
	CoT-SC (k = 16)	<u>85.02</u> (+4.50)
	CR (**ours**, n = 2)	**87.45** (**+6.93**)

Table 2: Results for various reasoning approaches on FOLIO-wiki-curated dataset.

Model	Method	Acc. ↑ (%)
-	[Random]	33.33
LLaMA-13B	Direct	49.13
	CoT	52.17 (+3.04)
	CoT-SC (k = 16)	<u>53.70</u> (+4.57)
	CR (**ours**, n = 2)	**55.87** (**+6.74**)
LLaMA-65B	Direct	74.78
	CoT	74.13 (-0.65)
	CoT-SC (k = 16)	<u>79.13</u> (+4.35)
	CR (**ours**, n = 2)	**79.57** (**+4.79**)
GPT-3.5-turbo	Direct	69.57
	CoT	<u>70.65</u> (+1.08)
	CoT-SC (k = 16)	69.32 (-0.25)
	CR (**ours**, n = 2)	**78.70** (**+9.13**)
GPT-4	Direct	89.57
	CoT	95.00 (+5.43)
	CoT-SC (k = 16)	<u>96.09</u> (+6.52)
	CR (**ours**, n = 2)	**98.04** (**+8.47**)

4.2 FOLIO wiki curated

The accuracy of 87.45% does not seem to be as competitive as human beings, so we carefully reviewed the FOLIO-wiki dataset. It turns out that many instances inside the dataset are problematic (see Appendix E.2 for a detailed list and examples shown in Appendix E.3).

Therefore, we removed all 74 such problematic instances, leaving the remaining 460 examples as a curated collection. The results in Table 2 indicate that the application of GPT-4 in conjunction with our method (CR) commands an astounding accuracy of 98.04% and maintains an error rate as minimal as 1.96%. This level of performance is almost twice as effective compared to the combination of GPT-4 and CoT-SC, which scored an accuracy of 96.09% and an error rate of 3.91%.

4.3 AutoTNLI

Experiment Setting. The AutoTNLI dataset, introduced by Kumar et al. [26], extends the INFOTABS dataset [18] to create a challenging Tabular Natural Language Inference (TNLI) task. This dataset, characterized by its complexity in natural language inference, comprises 1,478,662 table-hypothesis pairs labeled as either "Entail" or "Neutral" to signify the logical relationship between the table content and the hypothesis. In our approach, we interpret the tabular data as premises, similar to our application on the FOLIO dataset, ensuring a seamless adaptation of our Cumulative Reasoning (CR) methodology. We limit our evaluation to the first 1,000 table-hypothesis pairs due to the dataset's extensive size, employing two models, LLaMA-13B, and LLaMA-65B, and comparing the efficacy of Direct, Chain-of-Thought (CoT), CoT with Self-Consistency (CoT-SC), and our CR method.

Table 3: Results for various reasoning approaches on AutoTNLI dataset.

Model	Method	Acc. ↑ (%)
-	[Random]	50.00
LLaMA-13B	Direct	52.6
	CoT	54.1 (+1.5)
	CoT-SC (k = 16)	52.1 (-0.5)
	CR (**ours**, n = 4)	**57.0** (**+5.4**)
LLaMA-65B	Direct	59.7
	CoT	63.2 (+3.5)
	CoT-SC (k = 16)	61.7 (+2.0)
	CR (**ours**, n = 4)	**72.5** (**+12.8**)

Evaluation Results. The performance metrics, detailed in Table 3, underscore the significant advantage of the CR methodology over both CoT and CoT-SC, with LLaMA-65B showcasing a notable performance uplift of up to 9.3%. This enhancement not only demonstrates the effectiveness of CR in handling complex inference tasks but also highlights its superiority in leveraging the structural and linguistic nuances within the AutoTNLI dataset.

4.4 Game of 24

The Game of 24 is a mathematical puzzle where players aim to manipulate four given integers through basic arithmetic operations—addition, subtraction, multiplication, and division—to achieve a result of 24.

[Illustrative example for Game of 24]
- Numbers: [4, 5, 6, 10]
- Arithmetic Operations: $[+, -, \times, /, (,)]$
- **Solution**:

$$(10 - 6) * 5 + 4 = 24$$

Figure 4: An example from the Game of 24 dataset [63].

Settings and Baselines. To maintain consistency and fairness in our evaluation, we mirrored the experimental setup utilized by the Tree of Thoughts (ToT) [63]. We utilized a collection of 100 Game of 24 puzzles curated by ToT as our testbed. A puzzle is deemed successfully solved if the solution forms a valid equation that sums up to 24, utilizing each of the provided numbers exactly once. Our primary metric for evaluation is the accuracy rate, determined by the success rate across these 100 puzzles. In our comparative analysis, Cumulative Reasoning (CR) is benchmarked against various established prompting strategies, including standard Input-Output prompting (Direct), Chain-of-Thought prompting (CoT), and CoT-SC by aggregating the majority outcome from 100 sampled CoT trials (designated as k = 100), and Tree of Thoughts (ToT) with a breadth-first search width set at 5 (indicated as b = 5).

CR Setup. Within our CR algorithm, we maintain a set of "reached states", denoted by S. The initial state of S encompasses the start state s, representing the four input numbers devoid of any arithmetic operation. At each iteration, the algorithm randomly selects a state u from S. This state u is then provided to the Proposer, which in turn selects two numbers from within u and applies a basic arithmetic operation (addition, subtraction, multiplication, or division) to generate a new number, thus forming a new state v. To enhance efficiency, the Proposer is guided to avoid repeated operations.

Subsequently, the Verifier assesses the arithmetic operation proposed from u to v, ensuring its validity and potential to culminate in 24. Should the Verifier confirm the operation's legitimacy, v is incorporated into S. Upon identifying a state t that definitively achieves the target of 24, the Reporter articulates a solution tracing the pathway from s to t, culminating in the final answer.

The algorithm concludes either when the Reporter announces the final solution or when the iteration count surpasses a predefined limit, L, which is set to 50 for the purpose of our experiments.

Adhering to the methodology outlined by Yao et al. [63], our algorithm executes b parallel branches, with the evaluation criteria mandating that each input number is utilized precisely once. Due to the significant computational demands of GPT-4, our evaluation of the CR algorithm was confined to b values ranging from 1 to 5. The empirical results, as detailed in Table 4, demonstrate CR substantially outperforms ToT, showcasing an improvement margin of 24%—escalating from 74% to 98% accuracy—while engaging considerably fewer states. The average number of states traversed for ToT was derived from the experimental logs available in its official GitHub repository.

Comparison with ToT. In the specific context of the Game of 24, the methodologies of Cumulative Reasoning (CR) and Tree of Thoughts (ToT) share notable similarities yet diverge significantly in their approach to state generation and exploration. A fundamental difference lies in how each iteration processes: CR is designed to introduce a single new state at each step, focusing on a step-by-step progression towards the solution. Conversely, ToT is characterized by its generation of multiple candidate states during each iteration, employing a filtration mechanism to narrow down the feasible states. This operational distinction suggests that ToT engages in a broader exploration of potential, including invalid, states, compared to the more streamlined approach of CR.

Table 4: Results for various approaches on Game of 24 using GPT-4.

Method	Acc. ↑ (%)	# Visited states ↓
Direct	7.3	1
CoT	4.0	1
CoT-SC (k = 100)	9.0	100
Direct (best of 100)	33	100
CoT (best of 100)	49	100
ToT (b = 5)	74	61.72
CR (ours, b = 1)	84 (+10)	**11.68 (-50.04)**
CR (ours, b = 2)	94 (+20)	13.70 (-48.02)
CR (ours, b = 3)	97 (+23)	14.25 (-47.47)
CR (ours, b = 4)	97 (+23)	14.77 (-46.95)
CR (ours, b = 5)	**98 (+24)**	14.86 (-46.86)

Furthermore, ToT relies on a pre-defined search structure, utilizing a constant width and depth within its search tree. This rigid framework contrasts with CR's more dynamic strategy, where the language model (LLM) itself influences the depth of the search, adapting the exploration breadth as needed across different stages of the problem-solving process. Such flexibility in CR not only optimizes the search path but also tailors the exploration to the complexity and requirements of each specific problem, potentially enhancing efficiency and efficacy in reaching the correct solution.

5 Solving MATH Problems

5.1 CR without Code Environment

The MATH dataset [21] is a comprehensive benchmark designed to evaluate the mathematical reasoning capabilities of AI models. It spans a wide range of mathematical subdomains, including Algebra and Geometry, providing a robust framework for testing. Illustrative examples from the MATH dataset, alongside solutions generated by both CoT and our CR approach, can be found in Figures 5 and 6. In our experiments, we assessed the performance of Complex CoT and our method (CR), both with and without Progressive-Hint Prompting (PHP) [67]. For a fair evaluation, we reproduced the results of Complex CoT (w/ PHP) on a subset of 500 test examples, adhering to Lightman et al. [29], acknowledging the potential utilization of the remaining dataset portions in OpenAI's model training. This subset spans the full spectrum of difficulty levels, from the simplest (level 1) to the most challenging (level 5).

[Problem Description]

- Example ID: test/intermediate_algebra/1350.json
- Level: 5
- Subject: Intermediate Algebra
- **Problem**: Consider the polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

where the polynomial has integer coefficients and its roots are distinct integers.
Given $a_n = 2$ and $a_0 = 66$, the inquiry is to determine the least possible value of $|a_{n-1}|$.

[Ground Truth Solution]

- **Solution**: Since $f(x)$ has integer coefficients, the Integer Root Theorem asserts that any integer roots of $f(x)$ must divide the constant term $66 = 2 \cdot 3 \cdot 11$. Consequently, the potential integer roots of $f(x)$ are

$$\pm 1, \ \pm 2, \ \pm 3, \ \pm 6, \ \pm 11, \ \pm 22, \ \pm 33, \ \pm 66.$$

Additionally, given that all roots of $f(x)$ are integers, they are necessarily members of the aforementioned list.
We proceed to utilize Vieta's formulas. The roots of $f(x)$ yield a product of $(-1)^n \cdot \frac{a_0}{a_n}$, which evaluates to either 33 or -33. Simultaneously, the sum of these roots is $-\frac{a_{n-1}}{a_n} = -\frac{a_{n-1}}{2}$. To minimize $|a_{n-1}|$, we aim to reduce the absolute value of the root sum, ensuring that the product of the roots remains 33 or -33.
Considering two distinct scenarios:
Case 1: One of the roots is 33 or -33. In this scenario, the only other viable roots are ± 1. Here, the root sum's absolute value is at least 32.
Case 2: One root is 11 or -11, and another is 3 or -3. The only other plausible roots remain ± 1, leading to a root sum's absolute value of at least $11 - 3 - 1 = 7$. This is a more optimal outcome than Case 1. For an absolute root sum of 7, we find $|a_{n-1}| = 7|a_n| = 14$.
From our analysis, we deduce that $|a_{n-1}| \geq 14$. Verifying this, we observe that equality is attained by

$$\begin{aligned} f(x) &= 2(x+11)(x-3)(x-1) \\ &= 2x^3 + 14x^2 - 82x + 66, \end{aligned}$$

which is consistent with having integer coefficients and integer roots. Thus, the minimal achievable value for $|a_{n-1}|$ is $\boxed{14}$.

- **Answer**: **14**

Figure 5: An example from the MATH dataset [21, 29].

From Table 5, our method (CR) distinguishes itself by achieving significant advancements in performance across various mathematical subdomains, outperforming Complex CoT by a margin of 5.4%. Note that for our method (CR), we employed 4-shot prompting (4 examples for few-shot prompting) due to GPT-4's context length constraints (8k by default). The enhancements are particularly pronounced in the Number Theory, Probability, PreAlgebra, and Algebra categories compared to the Complex CoT approach (8-shot prompting). It is also evident that the PHP method further amplifies the performance of both Complex CoT and CR, establishing new state-of-the-art results with an overall accuracy of 58.0% using CR with PHP, with a margin of 4.2% over Complex CoT with PHP. Additionally, the "Iters" metric elucidates that CR, when synergized with PHP strategies, reaches self-consistent answers with fewer iterations.

[Chain-of-Thought Reasoning by GPT-4]

- **Reasoning**: "Let's think step by step:
  1. The roots of the polynomial are distinct integers, and the leading coefficient is 2, so the roots must be divisors of $\frac{66}{2} = 33$.
  2. The divisors of 33 are $\pm 1, \pm 3, \pm 11, \pm 33$.
  3. The sum of the roots of the polynomial is $-\frac{a_{n-1}}{a_n}$.
  4. To minimize $|a_{n-1}|$, we should select the roots to be as close to zero as possible.
  5. The smallest possible sum of the roots is $-1 - 3 + 11 + 33 = 40$,
  6. Thus, the smallest possible value of $|a_{n-1}|$ is $2 \times 40 = \boxed{80}$."
- **Answer**: **80** (**Wrong**)

[Cumulative-Reasoning by GPT-4]

- **Generated Hints**:
  1. "**Hint 1**: Recall Vieta's formulas, which relate the coefficients of a polynomial to the sums and products of its roots."
  2. "**Hint 2**: The product of the roots of the polynomial is equal to $-\frac{a_0}{a_n}$."
  3. "**Hint 3**: The sum of the roots of the polynomial is equal to $-\frac{a_{n-1}}{a_n}$."
  4. "**Hint 4**: Since the roots are distinct integers, consider the factors of $-\frac{a_0}{a_n}$ and their sums."
- **Generated Simple Questions and Answers**:

Question 1: "What is the product of the roots of the polynomial $f(x)$?"
Answer 1: "The product of the roots of the polynomial is $-\frac{a_0}{a_n} = -\frac{66}{2} = -33$."
Question 2: "What are the possible sets of distinct integer roots of the polynomial?"
Answer 2: "The possible sets of distinct integer roots are $\{-33, 1\}$, $\{-11, 3\}$, and $\{-3, 11, -1\}$."
Question 3: "What are the sums of the roots for each possible set of roots?"
Answer 3: "The sums of the roots are -32, -8, and 7 for the sets $\{-33, 1\}$, $\{-11, 3\}$, and $\{-3, 11, -1\}$, respectively."
Question 4: "What is the least possible value of $|a_{n-1}|$?"
Answer 4: "The least possible value of $|a_{n-1}|$ is $2 \times 7 = 14$."

- **Solution**: "Let's think step by step. Given that $a_n = 2$ and $a_0 = 66$, and the roots of the polynomial are distinct integers, the least possible value of $|a_{n-1}|$ is $\boxed{14}$."
- **Answer**: **14** (**Correct**)

Figure 6: Solutions for the example presented in Figure 5 from the MATH dataset, generated by CoT and CR. CoT will generate the answer directly through a chain of thought. By contrast, CR will first generate a few hints, then several simple and foundational questions, and then answer them by self, and finally conclude with the help of the generated hints and question-answer pairs.

As detailed in Table 5, CR substantially outperforms Complex CoT, demonstrating notable performance gains across multiple mathematical subdomains, with an improvement margin of 5.4%. This analysis employed a 4-shot prompting strategy, a necessity dictated by the contextual constraints of GPT-4-0314 (8k tokens). Remarkably, CR's performance enhancements were especially significant in areas such as Number Theory, Probability, PreAlgebra, and Algebra. Despite the limitations imposed by GPT-4's context length, the resilience and efficiency of CR are evident. The inclusion of PHP not only enhanced the performance of both Complex CoT and CR but also established new benchmarks, with CR and PHP reaching an unprecedented overall accuracy of 58.0%, thereby exceeding the Complex CoT with PHP by a margin of 4.2%. Moreover, the 'Iters' metric reveals that CR, when combined with PHP, achieves self-consistent solutions more efficiently.

Table 6 highlights the consistency of performance improvements across varying difficulty levels, underscoring the adaptability of the CR approach to diverse mathematical challenges. Notably, a 9.7% performance uplift at level 5 translates into a remarkable 43% relative improvement over the baseline Complex CoT method without PHP. This underscores CR's effectiveness in handling the most challenging problems in the dataset.

Table 5: Comparative performance on the MATH dataset using GPT-4 without code environment. We adopted a default temperature setting of $t = 0.0$, consistent with prior research settings (greedy decoding). PHP denotes the application of the progressive-hint prompting. "Iters" represents the average number of LLM interactions, and **Overall** reflects the overall results across MATH subtopics.

	w/ PHP	MATH Dataset (* denotes using 500 test examples subset following Lightman et al. [29])							
		InterAlgebra	Precalculus	Geometry	NumTheory	Probability	PreAlgebra	Algebra	**Overall**
CoT [40]	✗	-	-	-	-	-	-	-	42.50
Complex CoT, 8-shot [67]	✗	23.4	26.7	36.5	49.6	53.1	71.6	70.8	50.36
	✓	26.3	29.8	41.9	55.7	56.3	73.8	74.3	53.90
	(Iters)	3.2414	3.2435	3.2233	3.1740	2.8122	2.3226	2.4726	2.8494
Complex CoT* (repro., 8-shot)	✗	29.9	33.9	34.1	46.8	47.4	62.1	70.7	48.80
	✓	28.9	30.4	43.9	53.2	50.0	68.5	84.1	53.80
	(Iters)	2.7629	2.4643	2.7805	2.7581	2.4474	2.3780	2.5484	2.59
CR w/o code* (**ours**, 4-shot)	✗	28.9 (-1.0)	30.4 (-3.5)	39.0 (+4.9)	54.8 (+8.0)	57.9 (+10.5)	71.8 (+9.7)	79.3 (+8.6)	**54.20** (+5.40)
	✓	**32.0** (+3.1)	**35.7** (+5.3)	**43.9** (+0.0)	**59.7** (+6.5)	**63.2** (+13.2)	**71.8** (+3.3)	**86.6** (+2.5)	**58.00** (+4.20)
	(Iters)	2.6598	2.4821	2.5122	2.2903	2.2105	2.2195	2.3548	**2.40** (-0.19)

Table 6: Comparative performance on the MATH dataset using GPT-4 without code environment for different difficulty levels.

	w/ PHP	MATH Dataset (* denotes using 500 test examples subset)					
		Level 5	Level 4	Level 3	Level 2	Level 1	**Overall**
CoT [40]	✗	-	-	-	-	-	42.50
Complex CoT* (repro., 8-shot)	✗	22.4	38.3	62.9	72.2	79.1	48.80
	✓	23.9	43.8	63.8	86.7	83.7	53.80
CR w/o code* (**ours**, 4-shot)	✗	**32.1** (+9.7)	43.0 (+4.7)	62.9 (+0.0)	78.9 (+6.7)	83.7 (+4.6)	**54.20** (+5.40)
	✓	27.3 (+3.4)	**50.0** (+6.2)	**70.9** (+7.1)	**86.7** (+0.0)	**90.7** (+7.0)	**58.00** (+4.20)

5.2 CR with Code Environment

In this section, we extend Cumulative Reasoning (CR) with the inclusion of a code environment. Our experimental setup chooses not to utilize external aids such as memory modules, web browsing, or retrieval systems. Instead, we focus on a pure Python code environment to emulate a symbolic system. This approach aims to evaluate the LLM's intrinsic capabilities in computational problem-solving and logical reasoning. This involves a single reasoning context session without additional verifier LLMs.

In the CR framework with a code environment, the Python interpreter acts as a symbolic system that aids in verification. This setup allows for an intricate interplay between the proposer (LLM) and the verifier (LLM equipped with code environment). The LLM, acting as the proposer, can generate hypotheses, formulate mathematical expressions, and pose questions to itself. These steps are then executed and verified in the code environment, and the observations (outputs) are then interpreted by the LLM.

Our experimental results, as shown in Table 7 and Table 8, demonstrate the effectiveness of the CR methodology in a code environment. We compare our approach with PAL [16] and ToRA [17], two notable benchmarks in the field. CR with code significantly outperforms these methods, achieving an overall accuracy of 72.2% on the MATH dataset, achieving 38.9% relative improvement over PAL and 18.8% relative improvement over ToRA. More specifically, achieving 66.8% relative improvement of PAL, and 12.8% relative improvement over ToRA on the hardest level 5 MATH problems.

Table 7: Comparative performance on the MATH dataset using GPT-4 with Python code environment. We adopted a default temperature setting of $t = 0.0$, consistent with prior research settings (greedy decoding). "Sessions" denotes how many LLMs with a consecutive thinking context are involved in the reasoning process, and **Overall** reflects the overall results across MATH subtopics.

	# Sessions	MATH Dataset (* denotes 500 text examples subset)							
		InterAlgebra	Precalculus	Geometry	NumTheory	Probability	PreAlgebra	Algebra	**Overall**
PAL	-	32.8	29.3	38.0	58.7	61.0	73.9	59.1	51.8
PAL* (repro., 4 shot)	1	30.9	23.2	31.7	<u>66.1</u>	57.9	<u>73.2</u>	65.3	52.0
ToRA	-	40.0	37.2	44.1	68.9	67.3	82.2	75.8	61.6
ToRA* (repro., 4 shot)	1	<u>49.5</u>	<u>44.6</u>	<u>48.8</u>	49.5	<u>66.1</u>	67.1	<u>71.8</u>	<u>60.8</u>
CR w/ code* (**ours**, 2-shot)	1	**51.5** (+2.0)	**51.8** (+7.2)	**53.7** (+4.9)	**88.7** (+22.6)	**71.1** (+5.0)	**86.6** (+13.4)	**86.3** (+14.5)	**72.2** (+11.4)

Table 8: Comparative performance on the MATH dataset using GPT-4 and GPT-4-turbo with Python code environment for different difficulty levels.

	# Sessions	MATH Dataset (* denotes using 500 test examples subset)					
		Level 5	Level 4	Level 3	Level 2	Level 1	**Overall**
PAL	-	-	-	-	-	-	51.8
PAL* (repro., 4-shot)	1	31.3	45.3	60.0	65.6	<u>88.4</u>	52.0
ToRA	-	-	-	-	-	-	61.6
ToRA* (repro., 4-shot)	1	<u>46.3</u>	<u>53.9</u>	<u>69.5</u>	<u>75.6</u>	74.4	<u>60.8</u>
CR w/ code* (**ours**, 2-shot)	1	**52.2** (+5.9)	**66.4** (+12.5)	**81.9** (+12.4)	**90.0** (+14.4)	**90.7** (+2.3)	**72.2** (+11.4)

6 Related Work

Large Language Models. Language models have evolved into extremely large-scale neural networks [3, 9, 40–42, 44], which have shown impressive results across various tasks. GPT-3 [3] and its successors, such as Gopher [43], PaLM [6], GLaM [11], Chinchilla [22], Megatron–Turing NLG [47], LaMDA [51], OPT [65], LLaMA [52], PaLM 2 [1] and GPT-4 [40], have demonstrated that large auto-regressive language models can achieve high-quality results without extensive task-specific data collection or parameter updates.

Reasoning with Large Language Models (LLMs). The integration of reasoning capabilities in neural networks, through the generation of intermediate steps, has significantly advanced performance across various domains [20, 59, 60, 62, 64, 68]. Morishita et al. [39] enhance language models' reasoning by utilizing a synthetic corpus based on formal logic theory. Uesato et al. [54] provide a detailed comparison of process-based and outcome-based approaches in solving the GSM8K task, while Lightman et al. [29] contribute to the advancement of the field by curating the PRM-800K dataset, which offers step-by-step problem-solving guidance. Further, a considerable breadth of research focuses on augmenting reasoning through symbolic systems, such as code environments, knowledge graphs, and formal theorem provers, showcasing the utility of hybrid approaches in complex reasoning tasks [2, 4, 4, 10, 13, 16, 17, 23, 27, 30, 35, 37, 55, 56, 61].

Chain-of-Thought (CoT) Prompting. Initiated by Wei et al. [58], the CoT reasoning paradigm underscores the value of multi-step logical pathways in deriving conclusive answers. Building on this, Wang et al. [57] introduce self-consistency as an advanced decoding strategy, aiming to refine the basic greedy decoding used in CoT. Zhou et al. [68] and Khot et al. [25] further dissect complex tasks into manageable sub-tasks, optimizing the reasoning process. Creswell & Shanahan [8] explore the enhancement of reasoning quality through a beam search across reasoning traces, while Fu et al. [14] argue for increasing the complexity within few-shot prompts to improve performance. Recent developments include Li et al. [28]'s DIVERSE, which investigates various reasoning paths for the same question and employs a verifier for accuracy through weighted voting. Yao et al. [63]'s Tree-of-Thought (ToT) framework introduces deliberation in decision-making by considering multiple reasoning paths. Zheng et al. [67] propose an iterative approach, using previous responses as contextual clues in subsequent iterations. Feng et al. [12] highlight the theoretical and practical implications of CoT for solving complex real-world tasks, including dynamic programming.

7 Conclusion

In this work, we introduce Cumulative Reasoning (CR), a novel approach leveraging LLMs in a structured, iterative process that mirrors human cognitive strategies. By orchestrating the roles of proposer, verifier(s), and reporter, CR not only decomposes complex problems into manageable tasks but also effectively recomposes the validated steps into comprehensive solutions. This methodology has demonstrated superior performance across various domains, including logical inference, the Game of 24, and MATH problems, showcasing the versatility and potential of CR in advancing the capabilities of LLMs in complex problem-solving scenarios.

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[69] Pei Zhou, Jay Pujara, Xiang Ren, Xinyun Chen, Heng-Tze Cheng, Quoc V Le, Ed H Chi, Denny Zhou, Swaroop Mishra, and Huaixiu Steven Zheng. Self-discover: Large language models self-compose reasoning structures. *arXiv preprint arXiv:2402.03620*, 2024.

A Appendix for Prompts

The design of few-shot prompts is critical to guiding the behavior of each LLM role within CR. We crafted these prompts intending to encapsulate the essence of each role:

- The Proposer prompt encourages the generation of plausible next steps or hypotheses.
- The Verifier prompt focuses on assessing the validity of these propositions.
- The Reporter prompt aims at determining the sufficiency of information for concluding the reasoning process.

There have been several works [45, 66, 69] showing that zero-shot meta-prompts can also work well, which minimizes the bias introduced in the few-shot examples.

System: Suppose you are one of the greatest AI scientists, logicians, and mathematicians. Let us think step by step. Please use First-Order Logic (FOL) to deduce a “Proposition” from two given “Premises”. Please make sure that the “Proposition” is logically correct. Please ensure that the “Proposition" does not duplicate the “Premises”. Please make sure your reasoning is directly deduced from the “Premises” and “Propositions” rather than introducing unsourced common knowledge and unsourced information by common sense reasoning. Please remember that your “Proposition” should be useful to determine whether the Hypothesis is True, False, or Unknown.

User:
“Premises”: “{this.premises}”
We want to deduce more propositions to determine the correctness of the following Hypothesis:
“Hypothesis”: “{this.hypothesis}”

Assistant:
“Proposition”: “{[to be generated]}”

Figure 7: Prompt template for CR Proposer on logical inference tasks.

System: Suppose you are one of the greatest AI scientists, logicians, and mathematicians. Let us think step by step. Please use First-Order Logic (FOL) to determine whether the deduction of two given “Premises" to a “Proposition" is valid or not, and reply with True or False.

User:
“Premises": “{this.premises}”
“Proposition": “{this.proposition}”

Assistant:
“Judgement": “Is this deduction valid? {[True or False]}”

Figure 8: Prompt template for CR Verifier on logical inference tasks.

System: Suppose you are one of the greatest AI scientists, logicians, and mathematicians. Let us think step by step. Read and analyze the "Premises" first, then use First-Order Logic (FOL) to judge whether the "Hypothesis" is True, False, or Unknown. Please make sure your reasoning is directly deduced from the "Premises" and "Propositions" rather than introducing unsourced common knowledge and unsourced information by common sense reasoning.
User:
"Premises": "{this.premises}"
"Hypothesis": "{this.hypothesis}"
Assistant: "Thoughts": "Let us think step by step. From the premises, we can deduce propositions: {this.propositions}"
"Recall the Hypothesis": "{[this.hypothesis}"
"Judgement": "Now we know that the Hypothesis is {[True or False]}"

Figure 9: Prompt template for CR Reporter on logical inference tasks.

System: Suppose you are one of the greatest AI scientists, logicians, and mathematicians. You are very good at basic arithmetic operations. Use numbers and basic arithmetic operations (+ - * /) to obtain 24 with input numbers. In each step, You are only allowed to randomly choose arbitrary TWO of the input numbers to obtain a new number using arbitrary one basic arithmetic operation (AVOID duplicating with forbidden steps). Your calculation process must be correct.
User: Input: [a, b, c, d]
Next Step:
Assistant: c * d = e
User: Remaining Numbers:
Assistant: [a, b, e]

Figure 10: Prompt template for CR Proposer on Game of 24.

System: Suppose you are one of the greatest AI scientists, logicians, and mathematicians. You are very good at basic arithmetic operations. Use numbers and basic arithmetic operations (+ - * /) to obtain 24 with input numbers. Evaluate if the given intermediate step is correct and only use two existing numbers.
User:
Input: 10, 14
Intermediate step: 10 + 14 = 24
Assistant:
The intermediate step is valid.
Judgement:
Valid

User: Input: [a, b]
Intermediate step: [a op b = result]
Assistant:
{[reasoning to be generated]}
Judgement:
{[Valid or Invalid]}

Figure 11: Prompt template for CR Verifier (a) on Game of 24.

System: Suppose you are one of the greatest AI scientists, logicians, and mathematicians. You are very good at basic arithmetic operations. Use numbers and basic arithmetic operations (+ - * /) to obtain 24 with input numbers. Evaluate if given numbers can reach 24 (sure/likely/impossible).

User:
Input: 10, 14
Draft:

Assistant:
14 - 10 = 4
14 * 10 = 140
10 / 14 = 5/7
14 / 10 = 1.4
10 + 14 = 24

User:
Input: {remaining_numbers}
Draft:

Assistant:
sure
10 + 14 = 24

User:
{[reasoning to be generated]}
Judgement:
{[Valid or Invalid]}

Figure 12: Prompt template for CR Verifier (b) on Game of 24.

System: Suppose you are one of the greatest AI scientists, logicians, and mathematicians. You are very good at basic arithmetic operations. Use numbers and basic arithmetic operations (+ - * /) to obtain 24 with input numbers. You need to combine the given intermediate steps step-by-step into a complete expression.

User:
Input: 1, 1, 4, 6
Intermediate steps:
1 * 4 = 4 (left 1, 4, 6)
1 * 4 * 6 = 24

Assistant:
Draft:
Because 1 * 4 * 6 = 24, while 1 * 4 = 4. So 1 * (1 * 4) * 6 = 24.
Output:
1 * (1 * 4) * 6 = 24

User:
Input: {input}
Intermediate steps:
{intermediate steps}

Assistant:
Draft:
{[to be generated]}
Output:
{[to be generated]}

Figure 13: Prompt template for CR Reporter on Game of 24.

```
## Problem: {problem}

Solution: Let's think step by step.

### Preliminary Contents

- **Prelim 1**: {preliminary_contents_1}

- **Prelim 2**: {preliminary_contents_2}

- [...]

### Hints
- **Hint 1**: {useful_hints_1}

- **Hint 2**: {useful_hints_2}

- [...]

### Intermediate Steps: Question-AnswerSketch-Code-Output-Answer Pairs

Let's think step by step.

#### Question 1: {question_1}
- **Answer Sketch**: {answer_1}

##### Code for Question 1
{code_1}

#### Answer for Question 1
- **Answer**: {answer_1}

#### Question 2: [the second question you raised]
- **Answer Sketch**: [write a sketch of your answer to question 2]

##### Code for Question 2
{code_2}

#### Answer for Question 2
- **Answer**: {answer_2}

#### Question 3: {question_3}
- **Answer Sketch**: {answer_3}

##### Code for Question 3
{code_3}

#### Answer for Question 3
- **Answer**: {answer_3}

### Final Solution:

Recall the origin problem <MathP> {problem} </MathP>.

Let's think step by step.

#### Solution Sketch
{solution_sketch}

#### Code for Final Solution
{code_for_final_solution}

#### Final Answer
Final Answer: the answer is $\boxed{final_answer}$.
---
```

Figure 14: Prompt template for CR with code environment on solving MATH problems.

B More Experiments on Logical Inference Tasks

B.1 More experimental results

Table 9: Comparison results on LogiQA

Method	Acc. ↑	# Visited States ↓
Direct	31.69%	1
CoT	38.55%	1
CoT-SC	40.43%	**16**
ToT	<u>43.02</u>%	19.87
CR	**45.25**%	<u>17</u>

Table 10: Comparison results on ProofWriter

Method	Acc. ↑	# Visited States ↓
Standard	46.83%	1
CoT	67.41%	1
CoT-SC	69.33%	**16**
ToT	<u>70.33</u>%	24.57
CR	**71.67**%	<u>16.76</u>

Table 11: Comparison results on FOLIO-val

Method	Acc. ↑	# Visited States ↓
Standard	60.29%	1
CoT	67.65%	1
CoT-SC	68.14%	<u>16</u>
ToT	**69.12**%	19.12
CR	**69.11**%	**15.87**

Table 12: Comparison results on LD

Method	Acc. ↑	# Visited States ↓
Standard	71.33%	1
CoT	73.33%	1
CoT-SC	74.67%	**16**
ToT	<u>76.83</u>%	21.83
CR	**78.33**%	<u>16.98</u>

For a fair comparison of different methods on the LogiQA, ProofWriter, FOLIO (validation set), and LD datasets, we report the third-party reproduced results by Sun et al. [49], For implementation details on these experiments, please refer to their work.

B.2 Ablation studies

Table 13: Ablation studies on FOLIO wiki dataset using GPT-3.5-turbo model.

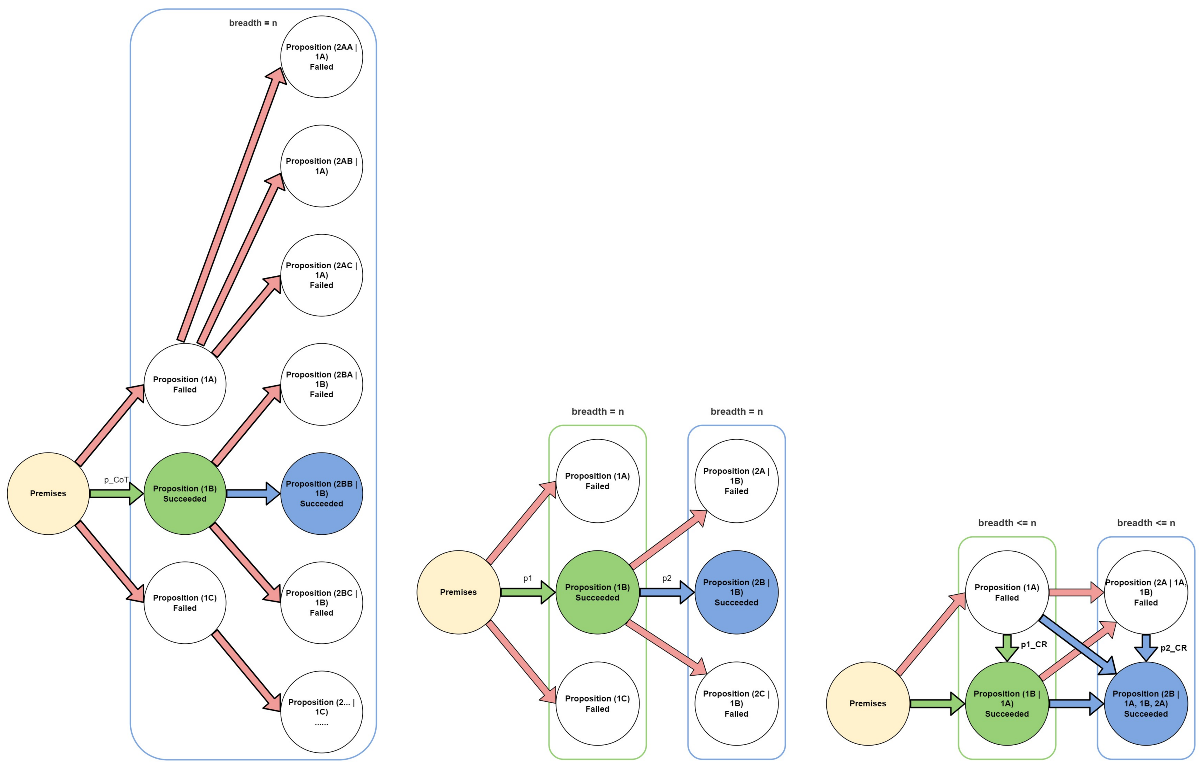
Model	Method	Acc. ↑ (%)
-	[Random]	33.33
GPT-3.5-turbo	Direct	62.92
	CoT	64.61 (+1.69)
	CoT-SC (k = 16)	63.33 (+0.41)
	CR (**ours**, n = 2)	**73.03** (**+10.11**)
	CR (**ours**, n = 2, w/o Verifier)	64.23 (+1.31)
	CR (**ours**, n = 2, w/o premises random choice)	<u>68.73</u> (<u>+5.81</u>)
	CR (**ours**, n = 2, w/o Verifier, w/o premises random choice)	67.23 (+4.31)

C Detailed Comparison of CoT, ToT and CR

To compare these methods, we consider a simple 2-stage reasoning process, which can be extended to multiple stages as well. For simplicity, whenever the model has a step-verifier, we assume that the verifier has 100% accuracy. Moreover, we assume that there exists exactly one correct reasoning path for the problem. We have the following definitions.

Definition C.1 (Arrival Probability)**.** For a given algorithm, we may compute its arrival probability as the probability of reaching the correct conclusion from the initial state, with one-experience successful invocation. Specifically, denote the arrival probability of CoT as P_{CoT}, the arrival probability of running CoT multiple times as $P_{\text{CoT-SC}}$, the arrival probability of ToT as $P_{\text{ToT}} = p_{1_{\text{ToT}}} p_{2_{\text{ToT}}}$, the arrival probability of CR as $P_{\text{CR}} = p_{1_{\text{CR}}} p_{2_{\text{CR}}}$. Here, $p_{1_{\text{ToT}}}$ and $p_{1_{\text{CR}}}$ are the probablity of getting the first reasoning step correctly, while $p_{2_{\text{ToT}}}$ and $p_{2_{\text{CR}}}$ are for the second step conditioned on the first step being correct.

Since both ToT and CR have verifiers, they can exclude the wrong reasoning path immediately, see Figure 15. Therefore, we immediately have $P_{\text{CoT}} \le p_{1_{\text{ToT}}} p_{2_{\text{ToT}}}$, as CoT explores more useless branches.



(a) CoT-SC (b) ToT (c) CR

Figure 15: Comparison between CoT-SC, ToT, and CR.

Notice that using $p_{1_{\text{CR}}}$ or $p_{2_{\text{CR}}}$ to denote the arrival probabilities of CR is not accurate, as CR will maintain a history of visited states. Therefore we use $p_{1_{\text{CR}|(\cdot)}}$ and $p_{2_{\text{CR}|(\cdot)}}$ to denote the probability conditioned with additional visited states. We have the following assumption.

Assumption C.2. $p_{1_{\text{ToT}}} \le p_{1_{\text{CR}}}$, $p_{2_{\text{ToT}}} \le p_{2_{\text{CR}}}$, In addition, $p_{1_{\text{CR}|(\cdot)}}$ and $p_{2_{\text{CR}|(\cdot)}}$ will monotonically increase as more nodes have been entered:

$$p_{1_{\text{ToT}}} \leq p_{1_{\text{CR}}|(\text{premises})} \leq p_{2_{\text{CR}}|(\text{premises,stage-1 node}_1)} \leq p_{2_{\text{CR}}|(\text{premises,stage-1 node}_1,\text{node}_2,\cdots,\text{node}_n)},$$

$$\begin{aligned} p_{2_{\text{ToT}}} &\leq p_{2_{\text{CR}}|(\text{premises,stage-1 nodes})} \leq p_{2_{\text{CR}}|(\text{premises,stage-1 nodes, stage-2 node}_1)} \\ &\leq p_{2_{\text{CR}}|(\text{premises,stage-1 nodes,stage-2 node}_1,\text{node}_2,\cdots,\text{node}_n)}, \end{aligned}$$

This assumption is natural and has been empirically validated in various tasks [36, 46] since CR will not enter the failed nodes multiple times, since the verifier has wiped out the possibilities of these nodes and their successors. The following lemma is handy for later comparison.

Lemma C.3. *For any positive integer n, for any probabilities $p_1 \in [0,1]$ and $p_2 \in [0,1]$, the following inequality holds:*

$$1 - (1 - p_1 \cdot p_2)^n \leq (1 - (1 - p_1)^n) \cdot (1 - (1 - p_2)^n). \tag{1}$$

Proof.

$$\begin{aligned} & 1 - (1 - p_1 \cdot p_2)^n \leq (1 - (1 - p_1)^n) \cdot (1 - (1 - p_2)^n) \\ \Leftrightarrow\ & 1 - (1 - p_1 \cdot p_2)^n \leq 1 - (1 - p_1)^n - (1 - p_2)^n + (1 - p_1)^n \cdot (1 - p_2)^n \\ \Leftrightarrow\ & (1 - p_1)^n + (1 - p_2)^n \leq (1 - p_1 \cdot p_2)^n + (1 - p_1)^n \cdot (1 - p_2)^n \\ \Leftrightarrow\ & (1 - p_1)^n + (1 - p_2)^n \leq (1 - p_1 \cdot p_2)^n + (1 - p_1 - p_2 + p_1 \cdot p_2)^n \end{aligned}$$

Notice that

$$(1 - p_1 \cdot p_2) + (1 - p_1 - p_2 + p_1 \cdot p_2) \equiv (1 - p_2) + (1 - p_2) \equiv 2 - p_1 - p_2,$$

WLOG, let $p_1 \geq p_2$, then

$$(1 - p_1 - p_2 + p_1 \cdot p_2) \leq (1 - p_1) \leq (1 - p_2) \leq (1 - p_1 \cdot p_2).$$

From the monotonicity of function $x^n + (2 - p_1 - p_2 - x)^n$ in the interval $(-\infty, \frac{2-p_1-p_2}{2}]$ and the interval $[\frac{2-p_1-p_2}{2}, +\infty)$ respectively, and the symmetry of $\{(1 - p_1 - p_2 + p_1 \cdot p_2), (1 - p_1 \cdot p_2)\}$ and the symmetry of $\{(1 - p_1), (1 - p_2)\}$ correspond to $y = \frac{2-p_1-p_2}{2}$, we conclude the proof. □

Theorem C.4 ($P_{\text{CoT-SC}} \leq P_{\text{ToT}} \leq P_{\text{CR}}$)**.** *Assume CoT-SC has n different trials, while ToT and CR search with breadth at most n. Under Assumptions C.2, the following inequality holds:*

$$P_{\textit{CoT-SC}} \leq P_{\textit{ToT}} \leq P_{\textit{CR}}. \tag{2}$$

Proof.

$$\begin{aligned} P_{\text{CoT-SC}} &\leq 1 - (1 - p_{\text{CoT}})^n \leq 1 - (1 - p_1 \cdot p_2)^n, \\ P_{\text{ToT}} &= (1 - (1 - p_1)^n) \cdot (1 - (1 - p_2)^n), \end{aligned}$$

Combined with Lemma C.3, now we have

$$P_{\text{CoT-SC}} \leq P_{\text{ToT}}.$$

From Assumption C.2, we have

$$P_{\text{ToT}} \le (1-(1-p_{1_{\text{CR}|(\text{premises})}})^n)\cdot(1-(1-p_{2_{\text{CR}|(\text{premises, stage-1 nodes})}})^n) \le P_{\text{CR}}.$$

Finally, we conclude that

$$P_{\text{CoT-SC}} \le P_{\text{ToT}} \le P_{\text{CR}}.$$

□

D More on Logic

Limitations of First-Order Logic Systems. It is not surprising that the labels verified by FOL are still not satisfying. There are several limitations inside the FOL systems:

1. Limitations of Expressiveness [33]: FOL even lacks the expressive power to capture some properties of the real numbers. For example, properties involving uncountably many real numbers often cannot be expressed in FOL. In addition, properties requiring quantification over sets of real numbers or functions from real numbers to real numbers cannot be naturally represented in FOL.

2. Translation Misalignment: Risk of semantic discrepancies during translation, rendering resolutions ineffective. For instance, translating statements as $\forall \text{Bird}(x) \Rightarrow \text{CanFly}(x)$ and $\forall x(\text{Fly}(x) \Rightarrow \text{Wings}(x))$ may cause a misalignment between "CanFly" and "Fly", leading to flawed conclusions. It often fails to capture the full richness and ambiguity of natural language and lacks basic common knowledge [15].

3. Undecidability: The general problem of determining the truth of a statement in FOL is undecidable [5, 53] (deeply connected to the halting problem), constraining its applicability for automated reasoning in complex tasks.

D.1 Illustrative example on higher-order logic

Here we present a refined example derived from the FraCas dataset to illustrate higher-order logic inference. It is noteworthy that the FraCas dataset [7] is dedicated to the realm of higher-order logic inference. This characterization also applies to a majority of the Natural Language Inference (NLI) datasets [26], which encompass their internal syntax, semantics, and logic. The intricate linguistic components such as quantifiers, plurals, adjectives, comparatives, verbs, attitudes, and so on, can be formalized with Combinatory Categorial Grammar (CCG) along with the formal compositional semantics [38].

Higher-order logic (HOL) has the following distinctive characteristics as opposed to FOL [38]:

Quantification over Functions: Higher-order logic (HOL) allows for lambda expressions, such as $\lambda y.\text{report\_attribute}(y, \text{report})$, whereby functions themselves become the subject of quantification. An illustration of this is found in the expression "a representative who reads this report." Here, quantification spans the predicates representing both the representative and the reading of the report, a phenomenon captured as a higher-order function. Unlike HOL, FOL is incapable of extending quantification to functions or predicates.

Generalized Quantifiers: The introduction of generalized quantifiers, such as "most," serves as another demarcation line between HOL and FOL. These quantifiers are capable of accepting predicates as arguments, enabling the representation of relations between sets, a feat that transcends the expressive capacity of FOL.

Modal Operators: Employing modal operators like "might" signifies a transition towards HOL. These operators, applicable to propositions, give rise to multifaceted expressions that defy easy reduction to the confines of FOL.

Attitude Verbs and Veridical Predicates: The integration of attitude verbs, such as "believe," and veridical predicates like "manage," injects an additional layer of complexity necessitating the use of HOL. These linguistic constructs can engage with propositions as arguments, interacting with the truth values of those propositions in subtle ways that demand reasoning extending beyond the capabilities of FOL.

Previously we have discussed the limitations of FOL systems, what about HOL systems? Crafting HOL programs that are solvable by symbolic systems is a daunting task, even for experts. It is also challenging for LLMs to write these intricate programs effectively. Using formal theorem provers based on higher-order (categorical) logic and (dependent) type theory ups the ante, making it even harder. However, CR solves these problems pretty well without resorting to and being restricted to symbolic systems, just like the way humans think.

[Modified Example FraCas-317]

- **Premises**:
  1. Most of the representatives who read the report have a positive attitude towards it.
  2. No two representatives have read it at the same time, and they may have different opinions about it.
  3. No representative took less than half a day to read the report.
  4. There are sixteen representatives.
- **Hypothesis**: It took the representatives more than a week to read the report, and most found it valuable.
- **Label**: **[True]**
- **Higher-Order Logic Premises**:
  1. $$\text{most}(\lambda x.\text{representative}(x) \land \text{reads}(x, \text{report}), \lambda x.\text{has\_positive\_attitude}(x, \text{report}))$$
  2. $$\begin{aligned} &\neg\exists x, y\, (x \neq y \land \text{representative}(x) \land \text{representative}(y) \land \\ &\qquad \text{read\_at\_same\_time}(x, y, \text{report})) \land \\ &\forall x, y\, (x \neq y \land \text{representative}(x) \land \text{representative}(y) \to \\ &\qquad \text{may}(\lambda z.\text{different\_opinions}(z, x, y, \text{report}))) \end{aligned}$$
  3. $\neg\exists x(\text{representative}(x) \land \text{took\_less\_than\_half\_day}(\lambda y.\text{read}(y, \text{report})))$
  4. $\text{count}(\lambda x.\text{representative}(x)) = 16$
- **Higher-Order Logic Hypothesis**:

$$\begin{aligned} &\exists t(\text{total\_reading\_time}(\text{report}) > 7 \land \text{time\_unit}(t, \text{day})) \land \\ &\quad \text{most}(\lambda x.\text{representative}(x), \lambda x.\text{found\_valuable}(x, \text{report})) \end{aligned}$$

[Cumulative-Reasoning by GPT-4]
- **Generated Propositions**:
  1. "Using P1 and P2, we can deduce Q1: Every representative read the report at a different time, and most representatives found the report valuable."
  2. "Using P3 and Q1, we deduce Q2: The total reading time must be at least 8 days. (since 16 representatives each took at least half a day)."
  3. "Using Q2 and P4, we deduce Q3: The representatives took at least 8 days to read the report."
- **Reasoning**: "Based on the intermediate propositions Q1, Q2, and Q3, the hypothesis that it took the representatives more than a week to read the report, and most found it valuable, is confirmed as True."
- **Prediction**: **[True]** (**Correct**)

E More on Datasets

E.1 More FOLIO Examples

[Example ID: 603]
- **Hypothesis**: Hyraxes lay eggs.
- **Premises**:
  1. The only types of mammals that lay eggs are platypuses and echidnas.
  2. Platypuses are not hyrax.
  3. Echidnas are not hyrax.
  4. No mammals are invertebrates.
  5. All animals are either vertebrates or invertebrates.
  6. Mammals are animals.
  7. Hyraxes are mammals.
  8. Grebes lay eggs.
  9. Grebes are not platypuses and also not echidnas.
- **Label**: **[False]**

[Chain-of-Thought Reasoning by GPT-4]
- **Reasoning**: "The premises state that the only types of mammals that lay eggs are platypuses and echidnas. Hyraxes are mammals but are neither platypuses nor echidnas. Since the conclusion is about hyraxes laying eggs but there's no direct information in the premises on this topic, the conclusion cannot be determined with certainty."
- **Prediction**: **[Unknown]** (**Wrong**)

[Cumulative-Reasoning by GPT-4]
- **Generated Propositions**:
  1. "All mammals are either vertebrates or invertebrates."
  2. "Hyraxes do not lay eggs, as they are neither platypuses nor echidnas."
- **Reasoning**: "We can deduce that the only types of mammals that lay eggs are platypuses and echidnas. Hyraxes are mammals, but they are neither platypuses nor echidnas. Therefore, hyraxes do not lay eggs."
- **Prediction**: **[False]** (**Correct**)

[Example ID: 546]

- **Hypothesis**: Extractive models are not always faithful.
- **Premises**:
  1. A summarization model is always faithful if it uses content from the input documents.
  2. Extractive models are a kind of summarization models.
  3. Extractive model can only use content from the input documents.
- **Label**: **[False]**

[Chain-of-Thought Reasoning by GPT-4]

- **Reasoning**: "The premises state that a summarization model is always faithful if it uses content from the input documents. Extractive models are a kind of summarization model, and they can only use content from the input documents. However, the premises do not provide any information about whether extractive models are always faithful or not."
- **Prediction**: **[Unknown]** (**Wrong**)

[Cumulative-Reasoning Reasoning by GPT-4]

- **Generated Propositions**:
  1. "All extractive models are faithful summarization models since they use content from the input documents."
  2. "All faithful summarization models use content from the input documents."
- **Reasoning**: "We can infer that extractive models are always faithful because they only use content from the input documents and satisfy the condition for a summarization model to be faithful. Therefore, the conclusion that extractive models are not always faithful is incorrect."
- **Prediction**: **[False]** (**Correct**)

[Example ID: 404]

- **Hypothesis**: No coursing is banned.
- **Premises**:
  1. Greyhound racing is a competitive sport where spectators bet on greyhounds.
  2. Coursing is a part of Greyhound racing.
  3. Some competitive sports where spectators bet on things are banned.
  4. Coursing involves spectators betting on a hare being pursued by greyhounds.
  5. Hares are small game.
  6. If a competitive sport involves spectators betting on small game, then it is banned.
- **Label**: **[False]**

[Chain-of-Thought Reasoning by GPT-4]

- **Reasoning**: "The premises state that coursing is a part of greyhound racing, and it involves spectators betting on a hare being pursued by greyhounds. Hares are small game, and if a competitive sport involves spectators betting on small game, then it is banned. It's also mentioned that some competitive sports where spectators bet on things are banned. However, the premises do not clearly state the status of coursing being banned or not banned."
- **Prediction**: **[Unknown]** (**Wrong**)

[Cumulative-Reasoning Reasoning by GPT-4]

- **Generated Propositions**:
  1. "If a competitive sport involves spectators betting on hares, then it is banned."
  2. "Coursing is a competitive sport that involves spectators betting on small game."
- **Reasoning**: "We know that coursing is a competitive sport involving spectators betting on small game, specifically hares. The premises also state that if a competitive sport involves spectators betting on small game, then it is banned. Thus, we can deduce that there is a possibility that coursing is banned. The hypothesis claims that no coursing is banned, which contradicts the deduction that coursing might be banned."
- **Prediction**: **[False]** (**Correct**)

[Example ID: 609]

- **Hypothesis**: Allison has room for broccoli.
- **Premises**:
  1. If you have room for dessert, you have room for broccoli.
  2. Everyone at Luis's dinner party has room for dessert, including Luis.
  3. Mauricia does not have room for broccoli.
  4. Luis's dinner party is the first ever dinner party that Allison has attended.
  5. Gustave has room for both broccoli and asparagus.
  6. Broccoli and asparagus are both vegetables.
- **Label**: **[True]**

[Chain-of-Thought Reasoning by GPT-4]

- **Reasoning**: "The premises state that if you have room for dessert, you have room for broccoli. Everyone at Luis's dinner party has room for dessert, including Luis. However, we do not have information specifically about Allison having room for dessert or not, even though it's her first dinner party. Thus, we can't conclude with certainty if Allison has room for broccoli."
- **Prediction**: **[Unknown]** (**Wrong**)

[Cumulative-Reasoning by GPT-4]

- **Generated Propositions**:
  1. "Gustave has room for vegetables."
  2. "Everyone at Luis's dinner party, including Mauricia, has room for dessert."
- **Reasoning**: "We know that:
  1. If one has room for dessert, they have room for broccoli.
  2. Everyone at Luis's dinner party, including Luis and Allison, has room for dessert.

  Using these premises, we can infer that since everyone at Luis's dinner party has room for dessert, they also have room for broccoli. Therefore, Allison, who is at Luis's dinner party, must have room for broccoli."
- **Prediction**: **[True]** (**Correct**)

E.2 Curating FOLIO wiki dataset

1. Missing common knowledge or contradictory to common knowledge; (9 in total, Example ID No. 34, 62, 162, 167, 228, 268, 526, 677, 679)
2. Overly ambiguous problems failing to provide unequivocal answers; (37 in total, Example ID No. 141, 215, 216, 223, 252, 261, 298, 321, 330, 396, 402, 409, 411, 431, 432, 456, 457, 482, 483, 496, 563, 572, 599, 624, 629, 641, 654, 660, 673, 682, 698, 750)
3. Inherent inconsistencies presented within the premises; (2 in total, Example ID No. 640, 643)

4. Vague premises or typographical errors; (2 in total, Example ID No. 314, 315)
5. Incorrect answers. (24 in total, Example ID No. 9, 46, 52, 84, 100, 144, 273, 276, 299, 310, 322, 345, 367, 437, 452, 453, 464, 557, 573, 578, 605, 632, 671, 715)

[Problem Description]

- Example ID: 679
- **Premises**:
  1. Zaha Hadid is a British-Iraqi architect, artist and designer.
  2. Zaha Hadid was born on 31 October 1950 in Baghdad, Iraq.
  3. Hadid was a visiting professor of Architectural Design at the Yale School of Architecture.
  4. Max is an aspiring architecture student, and he plans to apply to Yale School of Architecture.
- **Hypothesis**: Hadid was born in 1982.
- **FOL Label**: **[Unknown]**
- **Human Label**: **[False]**
- **Explanation**: *We can see that Zaha Hadid was born on 31 October 1950 in Baghdad, Iraq. This directly contradicts the hypothesis that Hadid was born in 1982. It is common knowledge that people are born only once, and someone can't be born in two different years.*

Figure 16: Example 679 from the FOLIO wiki dataset, the origin label provided by the FOL system is not correct, so we choose to curate this dataset, removing these examples with wrong labels. For more examples, please refer to Appendix E.3.

E.3 More examples on problems excluded from FOLIO wiki curated

Type 1 Error: Missing common knowledge or contradictory to common knowledge

[Example ID: 34]

- **Premises**:
  1. The Croton River watershed is the drainage basin of the Croton River.
  2. The Croton River is in southwestern New York.
  3. Kings are male.
  4. Water from the Croton River watershed flows to the Bronx.
  5. The Bronx is in New York.
- **Hypothesis**: Water from the Croton River flows to the Bronx.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 1: Missing common knowledge or contradictory to common knowledge in the premises]*
- **Explanation**: *We understand that the Croton River is in southwestern New York, and the Bronx is also located in New York. It is stated that water from the Croton River watershed flows to the Bronx, and the Croton River watershed is the drainage basin of the Croton River. It is common knowledge that water from a river flows to its drainage basin. Therefore, it is true that water from the Croton River flows to the Bronx.*

[Example ID: 268]

- **Premises**:
  1. Bernarda Bryson Shahn was a painter and lithographer.
  2. Bernarda Bryson Shahn was born in Athens, Ohio.
  3. Bernarda Bryson Shahn was married to Ben Shahn.
  4. People born in Athens, Ohio are Americans.
- **Hypothesis**: Bernarda Bryson Shahn was born in Greece.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 1: Missing common knowledge or contradictory to common knowledge in the premises]*
- **Explanation**: *We know that Bernarda Bryson Shahn was born in Athens, Ohio. It is common knowledge that Greece is not in Ohio. It also states that people born in Athens, Ohio, are Americans. Thus, it is false to conclude that Bernarda Bryson Shahn was born in Greece.*

[Example ID: 62]

- **Premises**:
  1. The Golden State Warriors are a team from San Francisco.
  2. The Golden State Warriors won the NBA finals.
  3. All teams attending the NBA finals have more than thirty years of history.
  4. Boston Celtics are a team that lost the NBA finals.
  5. If a team wins the NBA finals, then they will have more income.
  6. If a team wins or loses at the NBA finals, then they are attending the finals.
- **Hypothesis**: The Golden State Warriors will have more income for gate receipts.
- **Label**: **[True]**
- **Wrong Type**: *[Type 1: Missing common knowledge or contradictory to common knowledge in the premises]*
- **Explanation**: *We know that the Golden State Warriors won the NBA finals and that if a team wins the NBA finals, they will have more income. Therefore, we can infer that the Golden State Warriors will have more income. However, the hypothesis mentions 'more income for gate receipts,' and there is no information about gate receipts on the premises.*

Type 2 Error: Overly ambiguous problems failing to provide unequivocal answers

[Example ID: 496]

- **Premises**:
  1. Some fish may sting.
  2. Stonefish is a fish.
  3. It stings to step on a stonefish.
  4. Stonefish stings cause death if not treated.
  5. To treat stonefish stings, apply heat to the affected area or use an antivenom.
- **Hypothesis**: If you step on a stonefish and apply heat to the affected area, stings will cause death.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 2: Overly ambiguous problems failing to provide unequivocal answers]*
- **Explanation**: *The premises state that applying heat to the affected area or using antivenom can treat stonefish stings. Thus, if heat is applied to the affected area, it should help treat the sting and prevent death. However, it is not certain that applying heat to the affected area will prevent death, as it is possible that the sting is too severe to be treated with heat.*

[Example ID: 432]

- **Premises**:
  1. Vic DiCara plays guitar and bass.
  2. The only style of music Vic DiCara plays is punk music.
  3. Vic DiCara played in the band Inside Out.
- **Hypothesis**: If you step on a stonefish and apply heat to the affected area, stings will cause death.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 2: Overly ambiguous problems failing to provide unequivocal answers]*
- **Explanation**: *We know that Vic DiCara played in the band Inside Out and the only style of music he plays is punk music. This information implies that Inside Out played punk music while Vic DiCara was a member. However, it is not certain that Inside Out was a punk band, as it is possible that the band played a different style of music before Vic DiCara joined.*

[Example ID: 673]

- **Premises**:
  1. Cancer biology is finding genetic alterations that confer selective advantage to cancer cells.
  2. Cancer researchers have frequently ranked the importance of substitutions to cancer growth by P value.
  3. P values are thresholds for belief, not metrics of effect.
- **Hypothesis**: Cancer researchers tend to use the cancer effect size to determine the relative importance of the genetic alterations that confer selective advantage to cancer cells.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 2: Overly ambiguous problems failing to provide unequivocal answers]*
- **Explanation**: *We can deduce that cancer researchers tend to use P values, not effect sizes, to rank the importance of genetic alterations. Thus, the hypothesis contradicts the premises. However, it is still possible that cancer researchers use the cancer effect size to determine the relative importance of the genetic alterations that confer selective advantage to cancer cells.*

Type 3 Error: Inherent inconsistencies presented within the premises

[Example ID: 640]

- **Premises**:
  1. William Dickinson was a British politician who sat in the House of Commons.
  2. William Dickinson attended Westminster school for high school and then the University of Edinburgh.
  3. The University of Edinburgh is a university located in the United Kingdom.
  4. William Dickinson supported the Portland Whigs.
  5. People who supported the Portland Whigs did not get a seat in the Parliament.
- **Hypothesis**: William Dickinson did not get a seat in the Parliament.
- **Label**: **[True]**
- **Wrong Type**: *[Type 3: Inherent inconsistencies presented within the premises]*
- **Explanation**: *We have a contradiction. On one hand, we have information that William Dickinson supported the Portland Whigs, and people who supported the Portland Whigs did not get a seat in the Parliament. On the other hand, another premise states that William Dickinson was a British politician who sat in the House of Commons, which implies that he did get a seat in the Parliament.*

[Example ID: 643]
- **Premises**:
  1. William Dickinson was a British politician who sat in the House of Commons.
  2. William Dickinson attended Westminster school for high school and then the University of Edinburgh.
  3. The University of Edinburgh is a university located in the United Kingdom.
  4. William Dickinson supported the Portland Whigs.
  5. People who supported the Portland Whigs did not get a seat in the Parliament.
- **Hypothesis**: William Dickinson sat in the House of Commons.
- **Label**: **[True]**
- **Wrong Type**: *[Type 3: Inherent inconsistencies presented within the premises]*
- **Explanation**: *We have a contradiction. On one hand, we have information that William Dickinson supported the Portland Whigs, and people who supported the Portland Whigs did not get a seat in the Parliament. On the other hand, another premise states that William Dickinson was a British politician who sat in the House of Commons, which implies that he did get a seat in the Parliament.*

Type 4 Error: Vague premises or typographical errors

[Example ID: 314]
- **Premises**:
  1. Palstaves are a type of early bronze axe.
  2. Commonly found in northern, western and south-western Europe, palstaves are cast in moulds.
  3. John Evans is an archeologist who popularized the term "palstave".
  4. A paalstab is not an axe, but rather a digging shovel.
- **Hypothesis**: John Evans Popularized the term paalstab.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 4: Vague premises or typographical errors]*
- **Explanation**: *What is palstave and paalstab? Were they misspelled?*

[Example ID: 315]
- **Premises**:
  1. Palstaves are a type of early bronze axe.
  2. Commonly found in northern, western and south-western Europe, palstaves are cast in moulds.
  3. John Evans is an archeologist who popularized the term "palstave".
  4. A paalstab is not an axe, but rather a digging shovel.
- **Hypothesis**: There is an axe that is commonly found in Western Europe.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 4: Vague premises or typographical errors]*
- **Explanation**: *We can see that palstaves are a type of early bronze axe and they are commonly found in northern, western, and south-western Europe. Therefore, it is true that there is an axe that is commonly found in Western Europe. However, the premises also state that a paalstab is not an axe, but rather a digging shovel. Was paalstab the same thing as palstaves?*

Type 5 Error: Incorrect answers

[Example ID: 9]

- **Premises**:
  1. Palstaves are a type of early bronze axe.
  2. Pierre de Rigaud de Vaudreuil built Fort Carillon.
  3. Fort Carillon was located in New France.
  4. New France is not in Europe.
- **Hypothesis**: Fort Carillon was located in Europe.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 5: Incorrect answers]*
- **Explanation**: *We know that Fort Carillon was located in New France, and New France is not in Europe. Therefore, Fort Carillon was not located in Europe.*

[Example ID: 632]

- **Premises**:
  1. New York City is on the East Coast.
  2. Seattle is on the West Coast.
  3. If a person from a city on the East coast is traveling to a city on the west coast, they will be on a long flight.
  4. Most passengers on flights to Seattle from New York City are not in first class.
  5. People on long flights are uncomfortable unless they're in first class.
- **Hypothesis**: Some people flying from New York City to Seattle will be uncomfortable.
- **Label**: **[False]**
- **Wrong Type**: *[Type 5: Incorrect answers]*
- **Explanation**: *We can deduce the following: 1. A person traveling from New York City to Seattle will be on a long flight (since New York City is on the East Coast and Seattle is on the West Coast). 2. Most passengers on flights from New York City to Seattle are not in first class. 3. People on long flights are uncomfortable unless they're in first class. Given this information, we can conclude that some people flying from New York City to Seattle will be uncomfortable, as most of them are not in first class and long flights cause discomfort for those not in first class.*

[Example ID: 671]

- **Premises**:
  1. Westworld is an American science fiction-thriller TV series.
  2. In 2016, a new television series named Westworld debuted on HBO.
  3. The TV series Westworld is adapted from the original film in 1973, which was written and directed by Michael Crichton.
  4. The 1973 film Westworld is about robots that malfunction and begin killing the human visitors.
- **Hypothesis**: Michael Crichton has directed a film about robots.
- **Label**: **[Unknown]**
- **Wrong Type**: *[Type 5: Incorrect answers]*
- **Explanation**: *We can deduce that Michael Crichton wrote and directed the 1973 film Westworld, which is about robots that malfunction and begin killing the human visitors. Thus, it is true that Michael Crichton has directed a film about robots.*